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Orbits in one-dimensional finite linear cellular automata

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Periodicity and relaxation are investigated for the trajectories of the states in one-dimensional finite cellular automata with rules 90 and 150 [S. Wolfram, Rev. Mod. Phys. **55**, 601 (1983)]. The time evolutions are described with matrices. An eigenvalue analysis is applied to clarify the maximum value of period and relaxation.

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I. INTRODUCTION

Cellular automata are one of the simplest mathematical models for nonlinear dynamics to produce complicated patterns of behavior, which had been originally introduced by von Neumann [1]. Wolfram had reintroduced cellular automata as a model to investigate complexity and randomness [2]. He investigated many fundamental features of them [3–5]. Since then many authors have made efforts to clarify the properties of cellular automata and applied to natural systems [6].

One-dimensional cellular automata are described by the discrete time evolution of site a_i:

$$a_i(t+1) = F(a_{i-r}(t), a_{i-r+1}(t), \ldots, a_i(t), \ldots, a_{i+r}(t)), \qquad (1.1)$$

where a_i takes k discrete values over Z_k. The simplest model, *elementary cellular automaton*, consists of sites with two internal states over Z_2 interacting with the nearest-neighbor sites ($r = 1$). Wolfram introduced a naming scheme for these models and classified the behavior of cellular automata into four classes [2,3].

Most authors have worked on cellular automata within the scope of the infinite number of sites. A few works have concerned periodic boundary condition (cylindrical automata) [5,7–10]. In our previous paper [11] (referred to as paper I) we had investigated the periodic orbits of finite rule 90 cellular automata with Dirichlet boundary conditions. We analyzed the eigenvalue equations of the transfer matrices which describe the time evolution of the system. In the present paper the method is extended to the rule 150 case. Some results obtained in paper I will be cited again for completeness.

II. MODELS AND NUMERICAL RESULTS

We review the numerical results obtained in paper I on so-called rule 90 cellular automata following Wolfram's naming scheme. The time evolution of the ith site $a_i(t) \in \{0,1\}$ ($i = 1, \ldots, N$) is described as a sum modulo 2 of the nearest-neighbor sites:

$$a_i(t+1) = a_{i-1}(t) + a_{i+1}(t) \bmod 2. \qquad (2.1)$$

We use the Dirichlet boundary conditions $a_0 = a_{N+1} = 0$. In Wolfram's classification the rule 90 model belongs to the third class which shows the chaotic behavior. The time evolution is also expressed by the matrix

$$A(t+1) = UA(t), \qquad (2.2)$$

where $A(t) = {}^t(a_1(t), a_2(t), \ldots, a_N(t))$ describes the state (with N bits binary number) at t and the transfer matrix U is given by

$$U_{ij} = \begin{cases} 1 & j = i \pm 1 \\ 0 & \text{otherwise.} \end{cases} \qquad (2.3)$$

In paper I we had found the periodic structures of the transfer matrices U numerically as shown in Table I. They are summarized as follows:

$$U^{\Pi_N} = I \qquad (N \text{ is even}), \qquad (2.4a)$$

$$U^{\Pi_N + \pi_N} = U^{\pi_N} \qquad (N \text{ is odd}), \qquad (2.4b)$$

$$U^N = 0 \qquad (N = 2^n - 1). \qquad (2.4c)$$

Exponents Π_N and π_N are found numerically. Let us briefly describe how to find Π_N and π_N. First a sequence U^n ($n = 0, 1, 2, \ldots$) is produced. One can find that the sequence is periodic for $N \neq 2^n - 1$. For even N cases, the smallest value of n satisfying $U^n = I$ is found, where I is a unit matrix. The value n corresponds to Π_N in Eq. (2.4a). For odd N cases except $N = 2^n - 1$, there is no solution of $U^n = I$. One can, however, find the shortest value Π_N of period, $U^{\Pi_N + m} = U^m$. The minimum value of m stands for π_N in Eq. (2.4b).

For cellular automata with even number of sites [Eq. (2.4a)], every state except the null one (all sites are zero) is on the orbits with period not exceeding Π_N. The concrete periods depend on the initial states. The period Π_N corresponds to the least common multiple of them, and we call it the maximum period. The states on the orbits with the maximum period have the lowest symmetry. The states with some symmetries are on the orbits

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with shorter periods than Π_N. For cellular automata with odd number of sites except $N = 2^n - 1$ [Eq. (2.4b)], some states belong to the orbits with period Π_N or less and the others except the null state are drawn to the periodic orbits after some time steps not exceeding the maximum relaxation π_N. It is very interesting that in the $N = 2^n - 1$ $(n \in Z)$ case every state is drawn into the null state after at most N steps [Eq. (2.4c)]. In this case the configuration space has only one basin with the null state at the center. Schematic features of trajectories are shown in Fig. 1.

Next we investigate the rule-150 case, which also belongs to the third class in Wolfram's classification. The time evolution is described as a sum modulo 2 of the site itself and the nearest neighbors:

$$a_i(t+1) = a_{i-1}(t) + a_i(t) + a_{i+1}(t) \bmod 2. \qquad (2.5)$$

The transfer matrix is given by

$$U_{ij} = \begin{cases} 1, & j = i \pm 1 \text{ or } j = i \\ 0 & \text{otherwise.} \end{cases} \qquad (2.6)$$

The periodicity of the transfer matrices is also found numerically as shown in Table I. There are no cases drawn into the null state as described by Eq. (2.4c) for the rule 90 model. For cellular automata with $N = 3n + 2$ there appear the periodic orbits of period not exceeding Π_N

TABLE I. Periodicity of the transfer matrix U for rule-90 and rule-150 cellular automata.

N	rule 90	rule 150
3	$U^3 = 0$	$U^4 = I$
4	$U^6 = I$	$U^6 = I$
5	$U^5 = U$	$U^5 = U^4$
6	$U^{14} = I$	$U^{14} = I$
7	$U^7 = 0$	$U^8 = I$
8	$U^{14} = I$	$U^{16} = U^2$
9	$U^{13} = U$	$U^{62} = I$
10	$U^{62} = I$	$U^{62} = I$
11	$U^{11} = U^3$	$U^{12} = U^8$
12	$U^{126} = I$	$U^{42} = I$
13	$U^{29} = U$	$U^{28} = I$
14	$U^{30} = I$	$U^{32} = U^2$
15	$U^{15} = 0$	$U^{16} = I$
16	$U^{30} = I$	$U^{30} = I$
17	$U^{29} = U$	$U^{32} = U^4$
18	$U^{1022} = I$	$U^{1022} = I$
19	$U^{27} = U^3$	$U^{24} = I$
20	$U^{126} = I$	$U^{128} = U^2$
21	$U^{125} = U$	$U^{124} = I$
22	$U^{4094} = I$	$U^{4094} = I$
23	$U^{23} = U^7$	$U^{24} = U^{16}$
24	$U^{2046} = I$	$U^{2046} = I$
25	$U^{253} = U$	$U^{84} = I$
26	$U^{1022} = I$	$U^{1024} = U^2$
27	$U^{59} = U^3$	$U^{56} = I$
28	$U^{32766} = I$	$U^{32766} = I$
29	$U^{61} = U$	$U^{65} = U^5$
30	$U^{62} = I$	$U^{62} = I$
31	$U^{31} = 0$	$U^{32} = I$
32	$U^{62} = I$	$U^{64} = U^2$

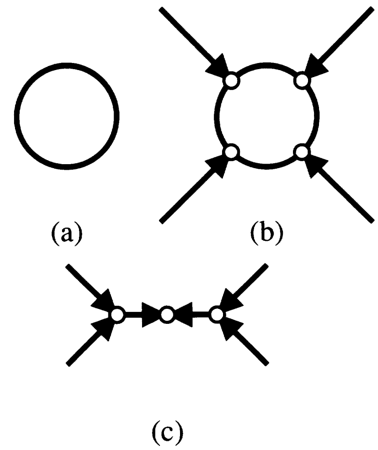

FIG. 1. Schematic features of the trajectories of cellular automata: (a) simple periodic orbit, (b) periodic orbit with relaxation, and (c) limit point.

with the relaxation path whose maximum length is π_N. For the others $N \neq 3n + 2$ all states are on the periodic orbits except the null state.

The maximum periods Π_N of the rule-90 and rule-150 cellular automata are found to coincide each other in many cases of the number of sites (see Table I and Figs. 2 and 3). Grassberger had reported the similar-

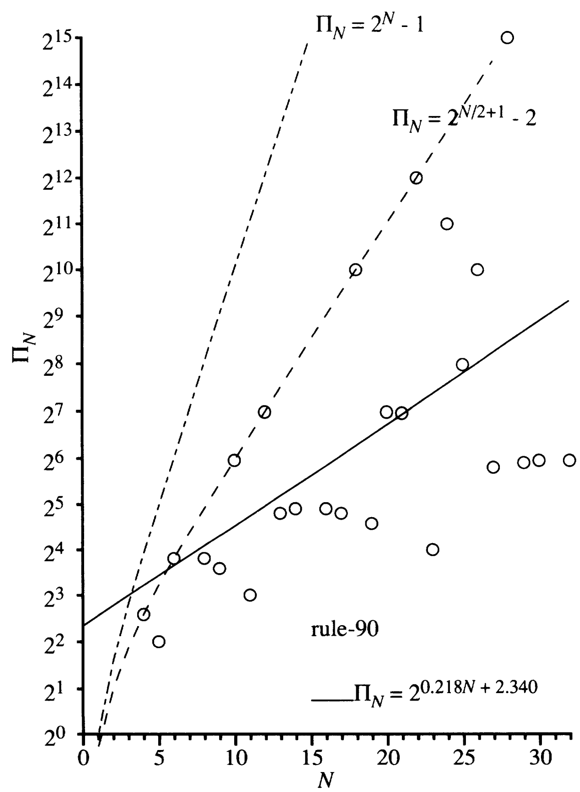

FIG. 2. Distribution of the maximum periods of the rule-90 cellular automata. The bold line is the average of the periods except $N = 2^n - 1$ cases. The dashed line denotes the curve $\Pi_N = 2^{N/2+1} - 2$, which fits peaks of N = 6, 10, 18, 22, and 28.

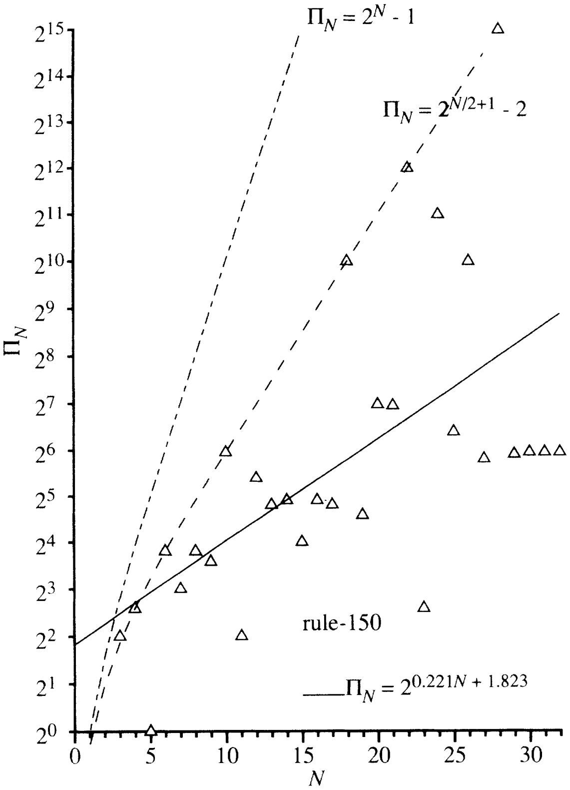

FIG. 3. Distribution of the maximum periods of the rule-150 cellular automata. The bold line is the average of the periods of all number of sites. The dashed line denotes the same as in Fig. 2.

ity between the rule-90 and rule-150 cases in behavior [12]. Especially the steepest peaks $N = 6, 10, 18, 22, 28$ coincide between the rule-90 and rule-150 cases, whose periods are expressed by $\Pi_N = 2^{N/2+1} - 2$. The maximum periods grow exponentially for the average of all numbers of sites except $N = 2^n - 1$ cases, in which all states are drawn into the null states within N steps for rule 90 and $\Pi_N = N + 1$ for rule 150.

III. EIGENVALUE ANALYSIS

The trajectories of rule-90 cylindrical automata had been investigated by Martin, Odlyzko, and Wolfram [5]. They analyzed *characteristic polynomials* which describe the states of cellular automata. The method seems to work well in the periodic boundary cases. To investigate the periodic structures analytically in the Dirichlet boundary cases, we had introduced the simpler method to analyze the eigenvalue polynomials in paper I. The eigenvalue equation of the transfer matrix

$$UA = -\lambda A \tag{3.1}$$

leads to the secular equation

$$D^N(\lambda) \equiv |U + \lambda I| = 0. \tag{3.2}$$

All calculations are carried out over Z_2 since a site a_i takes binary values.

The recursion relation of the eigenvalue polynomials for rule-90 cellular automata

$$D^N(\lambda) = \lambda D^{N-1}(\lambda) - D^{N-2}(\lambda) \tag{3.3}$$

gives the explicit form of $D^N(\lambda)$. The eigenvalue polynomial $D^N(\lambda)$ is over Galois field of order 2, GF(2), namely, all coefficients of λ^i are over Z_2:

$$D^N(\lambda) = \sum_{j=0}^{j_{\max}} \left(C_j^N \bmod 2\right) \lambda^{N-2j}, \tag{3.4}$$

where $j_{\max} = \lfloor N/2 \rfloor$, $\lfloor\ \rfloor$ is a Gaussian symbol ($\lfloor x \rfloor$ is the largest integer not exceeding x) and

$$C_j^N \equiv (-1)^j \binom{N-j}{j}. \tag{3.5}$$

Note that the definition of C_j^N is slightly different from that in paper I.

The eigenvalue polynomials for the rule-150 cellular automata are given by the replacement $\lambda \to \lambda + 1$ in Eqs. (3.3) and (3.4):

$$D^N(\lambda) = (\lambda + 1) D^{N-1}(\lambda) - D^{N-2}(\lambda), \tag{3.6}$$

TABLE II. Eigenvalue polynomials for rule-90 cellular automata.

N	$D^N(\lambda)$
3	λ^3
4	$\lambda^4 + \lambda^2 + 1$
5	$\lambda^5 + \lambda$
6	$\lambda^6 + \lambda^4 + 1$
7	λ^7
8	$\lambda^8 + \lambda^6 + \lambda^4 + 1$
9	$\lambda^9 + \lambda^5 + \lambda$
10	$\lambda^{10} + \lambda^8 + \lambda^4 + \lambda^2 + 1$
11	$\lambda^{11} + \lambda^3$
12	$\lambda^{12} + \lambda^{10} + \lambda^8 + \lambda^2 + 1$
13	$\lambda^{13} + \lambda^9 + \lambda$
14	$\lambda^{14} + \lambda^{12} + \lambda^8 + 1$
15	λ^{15}
16	$\lambda^{16} + \lambda^{14} + \lambda^{12} + \lambda^8 + 1$
17	$\lambda^{17} + \lambda^{13} + \lambda^9 + \lambda$
18	$\lambda^{18} + \lambda^{16} + \lambda^{12} + \lambda^{10} + \lambda^8 + \lambda^2 + 1$
19	$\lambda^{19} + \lambda^{11} + \lambda^3$
20	$\lambda^{20} + \lambda^{18} + \lambda^{16} + \lambda^{10} + \lambda^8 + \lambda^4 + \lambda^2 + 1$
21	$\lambda^{21} + \lambda^{17} + \lambda^9 + \lambda^5 + \lambda$
22	$\lambda^{22} + \lambda^{20} + \lambda^{16} + \lambda^8 + \lambda^6 + \lambda^4 + 1$
23	$\lambda^{23} + \lambda^7$
24	$\lambda^{24} + \lambda^{22} + \lambda^{20} + \lambda^{16} + \lambda^6 + \lambda^4 + 1$
25	$\lambda^{25} + \lambda^{21} + \lambda^{17} + \lambda^5 + \lambda$
26	$\lambda^{26} + \lambda^{24} + \lambda^{20} + \lambda^{18} + \lambda^{16} + \lambda^4 + \lambda^2 + 1$
27	$\lambda^{27} + \lambda^{19} + \lambda^3$
28	$\lambda^{28} + \lambda^{26} + \lambda^{24} + \lambda^{18} + \lambda^{16} + \lambda^2 + 1$
29	$\lambda^{29} + \lambda^{25} + \lambda^{17} + \lambda$
30	$\lambda^{30} + \lambda^{28} + \lambda^{24} + \lambda^{16} + 1$
31	λ^{31}
32	$\lambda^{32} + \lambda^{30} + \lambda^{28} + \lambda^{24} + \lambda^{16} + 1$

$$D^N(\lambda) = \sum_{j=0}^{j_{\max}} \left(C_j^N \bmod 2\right)(\lambda+1)^{N-2j},$$
$$= \sum_{k=0}^{N} \sum_{j=0}^{\lfloor (N-k)/2 \rfloor} \left(C'^N_{j,k} \bmod 2\right)\lambda^k, \quad (3.7)$$
$$C'^N_{j,k} \equiv (-1)^j \binom{N-j}{j}\binom{N-2j}{k}. \quad (3.8)$$

Some examples of $D^N(\lambda)$ for the rule-90 and rule-150 cellular automata are shown in Tables II and III.

These eigenvalue equations enable us to find the maximum periods Π_N and the maximum length π_N of the relaxation path to the periodic orbits. First we study nilpotent cases $D^N(\lambda) = \lambda^{P(N)}$. In paper I we found that those happen for the rule 90 with $N = 2^n - 1$ sites and $P(N) = N$. All states are drawn into the null state within steps N or less. The rule-150 model does not show similar behavior.

TABLE III. Eigenvalue polynomials for rule-150 cellular automata.

N	$D^N(\lambda)$
3	$\lambda^3+\lambda^2+\lambda+1$
4	$\lambda^4+\lambda^2+1$
5	$\lambda^5+\lambda^4$
6	$\lambda^6+\lambda^2+1$
7	$\lambda^7+\lambda^6+\lambda^5+\lambda^4+\lambda^3+\lambda^2+\lambda+1$
8	$\lambda^8+\lambda^6+\lambda^2$
9	$\lambda^9+\lambda^8+\lambda^5+\lambda^4+\lambda+1$
10	$\lambda^{10}+\lambda^4+1$
11	$\lambda^{11}+\lambda^{10}+\lambda^9+\lambda^8$
12	$\lambda^{12}+\lambda^{10}+\lambda^8+\lambda^4+1$
13	$\lambda^{13}+\lambda^{12}+\lambda^5+\lambda^4+\lambda+1$
14	$\lambda^{14}+\lambda^{10}+\lambda^8+\lambda^6+\lambda^2$
15	$\lambda^{15}+\lambda^{14}+\lambda^{13}+\lambda^{12}+\lambda^{11}+\lambda^{10}+\lambda^9+\lambda^8+\lambda^7$ $+\lambda^6+\lambda^5+\lambda^4+\lambda^3+\lambda^2+\lambda+1$
16	$\lambda^{16}+\lambda^{14}+\lambda^{10}+\lambda^8+\lambda^6+\lambda^2+1$
17	$\lambda^{17}+\lambda^{16}+\lambda^{13}+\lambda^{12}+\lambda^5+\lambda^4$
18	$\lambda^{18}+\lambda^{12}+\lambda^{10}+\lambda^8+\lambda^4+\lambda^2+1$
19	$\lambda^{19}+\lambda^{18}+\lambda^{17}+\lambda^{16}+\lambda^{11}+\lambda^{10}+\lambda^9$ $+\lambda^8+\lambda^3+\lambda^2+\lambda+1$
20	$\lambda^{20}+\lambda^{18}+\lambda^{16}+\lambda^{10}+\lambda^2$
21	$\lambda^{21}+\lambda^{20}+\lambda^9+\lambda^8+\lambda+1$
22	$\lambda^{22}+\lambda^{18}+\lambda^{16}+\lambda^8+1$
23	$\lambda^{23}+\lambda^{22}+\lambda^{21}+\lambda^{20}+\lambda^{19}+\lambda^{18}+\lambda^{17}+\lambda^{16}$
24	$\lambda^{24}+\lambda^{22}+\lambda^{18}+\lambda^8+1$
25	$\lambda^{25}+\lambda^{24}+\lambda^{21}+\lambda^{20}+\lambda^{17}+\lambda^{16}+\lambda^9+\lambda^8+\lambda+1$
26	$\lambda^{26}+\lambda^{20}+\lambda^{16}+\lambda^{10}+\lambda^2$
27	$\lambda^{27}+\lambda^{26}+\lambda^{25}+\lambda^{24}+\lambda^{11}+\lambda^{10}+\lambda^9$ $+\lambda^8+\lambda^3+\lambda^2+\lambda+1$
28	$\lambda^{28}+\lambda^{26}+\lambda^{24}+\lambda^{20}+\lambda^{16}+\lambda^{12}+\lambda^{10}+\lambda^8$ $+\lambda^4+\lambda^2+1$
29	$\lambda^{29}+\lambda^{28}+\lambda^{21}+\lambda^{20}+\lambda^{17}+\lambda^{16}+\lambda^{13}+\lambda^{12}+\lambda^5+\lambda^4$
30	$\lambda^{30}+\lambda^{26}+\lambda^{24}+\lambda^{22}+\lambda^{18}+\lambda^{14}+\lambda^{10}+\lambda^8$ $+\lambda^6+\lambda^2+1$
31	$\lambda^{31}+\lambda^{30}+\lambda^{29}+\lambda^{28}+\lambda^{27}+\lambda^{26}+\lambda^{25}+\lambda^{24}+\lambda^{23}$ $+\lambda^{22}+\lambda^{21}+\lambda^{20}+\lambda^{19}+\lambda^{18}+\lambda^{17}+\lambda^{16}+\lambda^{15}$ $+\lambda^{14}+\lambda^{13}+\lambda^{12}+\lambda^{11}+\lambda^{10}+\lambda^9+\lambda^8+\lambda^7$ $+\lambda^6+\lambda^5+\lambda^4+\lambda^3+\lambda^2+\lambda+1$
32	$\lambda^{32}+\lambda^{30}+\lambda^{26}+\lambda^{24}+\lambda^{22}+\lambda^{18}+\lambda^{14}+\lambda^{10}$ $+\lambda^8+\lambda^6+\lambda^2$

Next we discuss the cases that there are constant terms in polynomials. Those happen for the rule-90 with even number of sites since $C^N_{j_{\max}} = (-1)^{N/2-2}$ for even N. For the rule-150 case the polynomials have a constant term for $N \neq 3n+2$ by Eq. (3.6). The eigenvalue polynomials reduce to the simple forms as $\lambda^{P(N)} + 1 = 0$ by multiplying some power of λ and repeatedly substituting the eigenvalue equation to itself [13,14]. The minimum value of the power $P(N)$ corresponds to the maximum period Π_N. If the eigenvalue polynomial is factorized, we are able to get shorter periods depending on the initial state from those factors by the procedure mentioned above. For instance, we consider $N = 4$ rule-150 cellular automata (Fig. 4). The eigenvalue equation $\lambda^4+\lambda^2+1=0$ reduces to $\lambda^6+1=0$ and the maximum period is given as $\Pi_4 = 6$. The eigenvalue equation is also factorized to $(\lambda^2+\lambda+1)^2 = 0$. We find another solution $\lambda^3 = 1$ from $\lambda^2+\lambda+1=0$. Therefore there are period-6 and period-3 orbits for $N = 4$ rule-150 cellular automata.

Explicit expression of the maximum period Π_N is simply obtained for $N = 2^n - 1$ rule-150 cellular automata. In this case the identity $\binom{N-j}{j} \bmod 2 = 0$ holds for $j \neq 0$. The nonzero elements of $C'^N_{j,k}$ are $C'^N_{0,k} = \binom{N}{k}$. By the identity $\binom{2^n-1}{k} \bmod 2 = 1$ for all non-negative integers k, the eigenvalue polynomial reduces to $D^N(\lambda) = \sum_{k=0}^{N}\lambda^k$. Following the above procedure, we find the maximum period $\Pi_N = N+1$.

Finally we study the cases that the lowest powers of polynomials are greater than 0, namely, they have the forms as $D^N(\lambda) = \lambda^{p(N)}f(\lambda)$, where $f(\lambda)$ is a polynomial with a constant term. The number of null solutions, $p(N)$, corresponds to the maximum length of relaxation path π_N. Applying the above procedure to $f(\lambda)$, we obtain the maximum period and shorter ones. For example, we show the case of $N = 5$ rule 90 cellular automata (Fig. 5). The eigenvalue polynomial is $D^N(\lambda) = \lambda(\lambda^4+1)$. The maximum length of relaxation path is $\pi_5 = 1$. From $\lambda^4+1=0$, we find the maximum

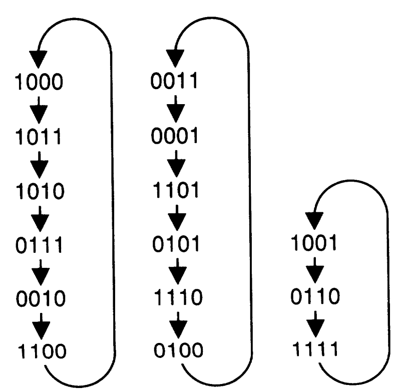

FIG. 4. The orbits of $N = 4$ rule-150 cellular automaton. All states are classified into three orbits excepts the null states. Two of them are period-6 and the other is period-3. The null state is an isolated fixed point. The periods are the same as those for rule 90, although the detail behaviors are not (see Fig. 1 in paper I).

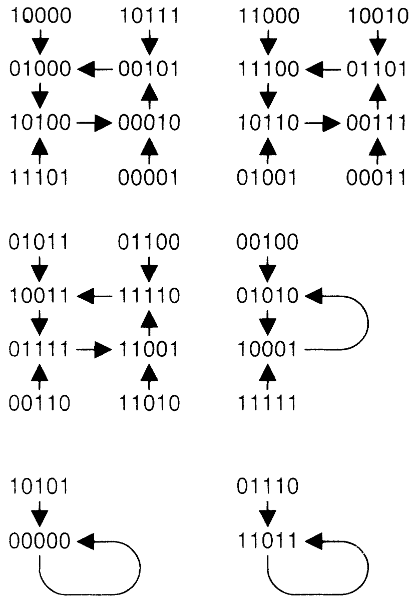

FIG. 5. The orbits of $N = 5$ rule 90 cellular automaton. This figure is the same as Fig. 2 in paper I.

period $\Pi_5 = 4$. We also find shorter periods-2 and -1 by factorizing $\lambda^4 + 1 = (\lambda^2+1)^2 = (\lambda+1)^4$.

A simple recursion relation of the maximum period and maximum length of relaxation are found for rule 90 cellular automata with odd number of sites. The new recursion relation

$$D^N(\lambda) = \lambda^2 D^{N-2}(\lambda) - D^{N-4}(\lambda) \tag{3.9}$$

holds by virtue of algebra in GF(2). This gives a simple form $D^{2n+1}(\lambda) = \lambda D^n(\lambda^2)$. The maximum period and the maximum length of relaxation path of $N = 2n+1$ rule-90 cellular automata are described by those of $N = n$ sites as

$$\Pi_{2n+1} = 2\Pi_n + 1, \tag{3.10}$$

$$\pi_{2n+1} = 2\pi_n + 1. \tag{3.11}$$

IV. CONCLUDING REMARKS

We investigate the periodic structures of one-dimensional rule-90 and rule-150 cellular automata with Dirichlet boundary conditions. We find three types of behavior. The first is a periodic one which appears in cellular automata with an even number of sites for rule 90 and $N \neq 3n+2$ for rule 150. The second appears in the cases of an odd number of sites for rule 90 and $N = 3n+1$ for rule 150. There are some periodic orbits and irreversible relaxation paths to them. The peculiar behavior happens to the case $N = 2^n - 1$ for rule 90. All states are drawn into the null state within N steps.

The eigenvalue equations of the transfer matrices are analyzed. The maximum period is obtained by finding the minimum power to satisfy $\lambda^{P(N)} + 1 = 0$. Shorter periods are given by factorizing the eigenvalue polynomials. The number of null solutions of the polynomials gives the maximum length of relaxation path. For some special cases, we find the explicit forms of the maximum period. The distribution of periods and relaxation is still not clear.

Roots of the eigenvalue equations, in general, are found over the Galois extension of finite fields [13,14]. For all positive integers N there are primitive polynomials $\mathcal{P}(x)$ of degree N over GF(2). One of the roots α of $\mathcal{P}(x)$ generates the Galois extension GF(2^N), whose elements are $\{0, 1, \alpha, \alpha^2, \ldots, \alpha^{2^N-2}\}$ and $\alpha^{2^N-1} = 1$. Other roots of $\mathcal{P}(x)$ are called *conjugate* of α and generate the *isomorphic* Galois extensions. A general polynomial of degree N has roots over GF(2^N). A eigenvalue polynomial has N roots $\{a_i\} \in$ GF(2^N) ($i = 1, 2, \ldots, N$). For instance, in the $N = 4$ rule-150 case, the equation $\lambda^4 + \lambda^2 + 1 = 0$ has roots in $\{0, 1, \alpha, \alpha^2, \ldots, \alpha^{14}\}$ ($\alpha^{15} = 1$), where α is one of roots of the forth degree primitive polynomial $\lambda^4 + \lambda + 1 = 0$. Explicitly the roots are α^5 and α^{10} with multiplicity 2 for each. The minimum common multiplier of the roots, which satisfies $\alpha^k = 1$, is α^{30}. This seems to suggest the maximum period-6, namely, $(\alpha^5)^6 = (\alpha^{10})^3 = \alpha^{30} = 1$. This procedure, however, does not work for other cases. For example, the eigenvalue polynomial of the $N = 3$ rule-150 case is $\lambda^3 + \lambda^2 + \lambda + 1$. It can be factorized to $(\lambda+1)^3$ and the root is $\lambda = 1$ with multiplicity 3. The periods, however, are 4, 2 and 1. More number theoretical studies are expected.

The elementary cellular automata are also subjects to build a built-in self-test of very-large-scale integrated circuits [15,16]. Usually the shift registers, which generate pseudorandom sequences of length $2^N - 1$ with N registers, are used for built-in self-tests. Our results show that the elementary cellular automata can also generate exponentially long but not maximum sequences. By a fine-tuned mixture of rule-90 and rule-150 cases, *hybrid cellular automata*, it has been shown to be able to produce maximum length pseudo-random sequences.

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